

# The Linear Correlation Coefficient vs. the Cross Term in Bose-Einstein Correlations

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We investigate the nature of the new cross term for Gaussian parameterizations of Bose-Einstein correlations of identical particles emitted from purely chaotic hadron sources formed by relativistic heavy ion collisions. We find that this additional parameter in the so-called Bertsch parameterization can be expressed in terms of a linear “out-longitudinal” correlation coefficient for emission of bosons and two already known “radius” parameters,  $R_l$  and  $R_o$ . The linear correlation coefficient is of kinematical nature and can be used to determine the widths of longitudinal momentum distributions.

In the ongoing search for the quark-gluon plasma (QGP) one is especially interested in the volumes and the lifetimes of the hot and dense zones of nuclear matter, *i.e.*, the fireballs that are generated in relativistic heavy-ion collisions. A tool to measure space-time extensions [1] of the fireball is given through the Hanbury-Brown/Twiss (HBT) interferometry [2], which provides the measurement of Bose-Einstein correlations (BEC) [3].

BEC functions of identical bosons are quantum-statistical observables which contain information about ten quantities [4] which characterize a hadron source: lifetimes, longitudinal and transverse extensions of the chaotic and the coherent source, temporal and spatial (longitudinal and transverse) coherence lengths and the chaoticity. In the case of pion interferometry it has been argued that a possible coherent subcomponent of the hadron source is almost unobservable [5], because resonance decays contribute strongly to BEC. Many parameterizations of experimental BEC functions are based on a neglect of a coherent component thereby resulting in the description of the hadron-emitting source with only three quantities, (*e.g.*, lifetime, longitudinal and transverse radii of the purely chaotic hadron source).

For heavy-ion collisions a very popular parameterization of purely chaotic BEC functions is provided through the so-called Bertsch parameterization [6] wherein a two-particle BEC function is parameterized in terms of the three momentum-dependent “radius” parameters  $R_l(\vec{K})$ ,  $R_s(\vec{K})$ , and  $R_o(\vec{K})$ . Here,  $\vec{K} \equiv \frac{1}{2}(\vec{k}_1 + \vec{k}_2)$  is the aver-

age momentum of two identical bosons which are emitted with their individual momenta,  $\vec{k}_1$  and  $\vec{k}_2$ , respectively. In the HBT Cartesian coordinate system the “longitudinal” or  $z$  direction (subscript  $l$ ) is parallel to the beam, the “out” or  $x$  direction (subscript  $o$ ) is parallel to the component of  $\vec{K}$  which is perpendicular to the beam, and the “side” or  $y$  direction (subscript  $s$ ) is the remaining direction. In ref. [7] it was shown that even in the case of a purely chaotic hadron source, three quantities are not adequate to fully parameterize the system under consideration. *E.g.*, a cylindrically symmetric expanding fireball has in general to be described with an additional “cross-term” radius<sup>1</sup>,  $R_{ol}(\vec{K})$ .

The generalized (Gaussian) Bertsch parameterization of a purely chaotic two-particle Bose-Einstein correlation function is given through [7]

$$C_2(\vec{k}_1, \vec{k}_2) = 1 + \lambda(\vec{K}) \cdot \exp[-q_l^2 R_l^2(\vec{K}) - q_o^2 R_o^2(\vec{K}) - q_s^2 R_s^2(\vec{K}) - 2 q_o q_l R_{ol}^2(\vec{K})]. \quad (1)$$

In eq. (1) the  $q_i$  ( $i = l, s, o$ ) refer to the components of the momentum difference  $\vec{q} = \vec{k}_1 - \vec{k}_2$ , and  $\lambda(\vec{K})$  is the momentum dependent incoherence factor which accounts for reductions of the BEC due to long-lived resonances [5] and averaging due to phase-space, respectively.

In a physical interpretation one is troubled by the fact that one and the same two-particle BEC function is described in case (a) of the generalized Bertsch parameterization through four “radius” parameters ( $R_l$ ,  $R_s$ ,  $R_o$ ,  $R_{ol}$ ), and (b) in case of the Yano-Koonin-Podgoretskii (YKP) parameterization [9] through three “radius” parameters ( $R_0$ ,  $R_l$ ,  $R_\perp$ ) and one kinematic quantity, the Yano-Koonin velocity ( $v$ ). It is the purpose of this paper to take a closer look at the recently reported new cross term,  $R_{ol}$ , and to investigate its nature.

Before we begin our analysis, we recall that the four “radius” parameters of the generalized Bertsch parameterization can be expressed in terms of variances of the (boosted) components of the space-time points  $x^\nu = (t, x, y, z) \in \Sigma$  ( $\Sigma$  is the freeze-out hypersurface) and covariances between the components of  $x^\nu$  (*cf.* ref. [1])

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<sup>1</sup>Experimentally, a “cross-term” has been observed, *e.g.*, by the NA35/NA49 Collaborations [8].

$$\begin{aligned} R_s^2(\vec{K}) &\approx \sigma_y^2, & R_o^2(\vec{K}) &\approx \sigma_{x-\beta_{\perp}t}^2, \\ R_l^2(\vec{K}) &\approx \sigma_{z-\beta_{\parallel}t}^2, & R_{ol}^2(\vec{K}) &\approx \sigma_{x-\beta_{\perp}t, z-\beta_{\parallel}t}. \end{aligned} \quad (2)$$

In eq. (2) we use  $\beta_i = K_i/E_K$  ( $i = \parallel, \perp$ ), where  $E_K = \sqrt{m^2 + \vec{K}^2}$ .

Let us now consider a quite general mathematical representation of a two-dimensional Gaussian,  $f(q_l, q_o)$ , in the two variables  $q_l$  and  $q_o$  with fixed parameters  $R_l$ ,  $R_o$ , and  $\lambda$

$$f(q_l, q_o) = 1 + \lambda \cdot \exp[-q_l^2 R_l^2 - q_o^2 R_o^2 + 2 q_o q_l R_o R_l \rho_{ol}], \quad (3)$$

where  $\rho_{ol}$  is the “linear correlation coefficient” which gives the strength of linear correlation between the two quantities  $q_l R_l$  and  $q_o R_o$ . A comparison between eq. (1) and eq. (3) yields for the “cross-term” (while ignoring the “side-term” without loss of generality)

$$R_{ol}^2(\vec{K}) \equiv -\rho_{ol}(\vec{K}) \cdot R_o(\vec{K}) R_l(\vec{K}), \quad (4)$$

or in terms of the variances and covariances for the (boosted) “longitudinal” and “out” components of  $x^\nu$

$$\rho_{ol}(\vec{K}) \equiv -\frac{R_{ol}^2(\vec{K})}{R_o(\vec{K}) R_l(\vec{K})} \approx -\frac{\sigma_{x-\beta_{\perp}t, z-\beta_{\parallel}t}}{\sqrt{\sigma_{x-\beta_{\perp}t}^2 \sigma_{z-\beta_{\parallel}t}^2}}, \quad (5)$$

with  $|\rho_{ol}(\vec{K})| \leq 1 \forall \vec{K}$ , representing the Cauchy-Schwarz inequality relation for  $\rho_{ol}$ .

Thus the “cross-term” radius parameter,  $R_{ol}$  in the Bertsch parameterization can be expressed in terms of a linear “out-longitudinal” correlation coefficient,  $\rho_{ol}$ , for emission of bosons and the two already known “radius” parameters,  $R_l$  and  $R_o$ .

In order to illustrate the nature of the cross-term, in Fig.1 we plot the pion rapidity distribution (top) and the scatter plot  $q_l$  vs.  $q_o$  (bottom). These plots are the results of the microscopic transport model RQMD [10] calculations for 200 AGeV S+S central collisions. All plots are generated in the nucleon-nucleon center of mass system; the rapidity windows are labeled in the figure.

In the mid-rapidity window no correlation between  $q_l$  and  $q_o$  is seen while sizable correlations are found in other rapidity windows. In addition the strength of the correlation depends on the rapidity for which one constructs the scatter plot<sup>2</sup>. Qualitatively, the strength of

the correlation is consistent with the NA35/NA49 result [8]. No correlation is seen between  $q_l$  and  $q_s$  or  $q_s$  and  $q_o$  in this model calculation. As discussed later in more detail, this longitudinal momentum dependence implies that the cross-term is kinematic in nature. However, as can be seen in the figure this dependence could be used to determine the rapidity width of the particles under study.

For the sake of further illustration we consider results of a hydrodynamical analysis [11,12] of 160 AGeV  $Pb + Pb$  (central) collisions using the HYLANDER code [13]. In the following we discuss the fireball emission of (identical)  $K^-K^-$  pairs because in ref. [1] it has been argued that for  $K^-K^-$  BEC functions their numerical calculation and subsequent fit can be replaced by the direct evaluation of eqs. (2) in order to determine the four “radius” parameters of the generalized Bertsch parameterization. We perform our analysis for average, transverse kaon pair momenta  $K_\perp = 1.5$  GeV/c (an analysis for different average transverse kaon pair momenta,  $K_\perp \neq 0$ , leads qualitatively to the same results).

We now introduce the strengths of linear correlation among  $t$ ,  $x$ , and  $z$

$$\rho_{t,x} \equiv \frac{\sigma_{t,x}}{\sqrt{\sigma_t^2 \sigma_x^2}}, \quad \rho_{t,z} \equiv \frac{\sigma_{t,z}}{\sqrt{\sigma_t^2 \sigma_z^2}}, \quad \rho_{x,z} \equiv \frac{\sigma_{x,z}}{\sqrt{\sigma_x^2 \sigma_z^2}}. \quad (6)$$

From eqs. (2) and (6) we are now able to rewrite  $R_l$ ,  $R_o$ , and  $R_{ol}$

$$\begin{aligned} R_o^2(\vec{K}) &\approx \sigma_x^2 + \beta_{\perp}^2 \sigma_t^2 - 2 \beta_{\perp} \sigma_t \sigma_x \rho_{t,x}, \\ R_l^2(\vec{K}) &\approx \sigma_z^2 + \beta_{\parallel}^2 \sigma_t^2 - 2 \beta_{\parallel} \sigma_t \sigma_z \rho_{t,z}, \\ R_{ol}^2(\vec{K}) &\approx \sigma_x \sigma_z \rho_{x,z} + \beta_{\parallel} \beta_{\perp} \sigma_t^2 \\ &\quad - \beta_{\parallel} \sigma_t \sigma_x \rho_{t,x} - \beta_{\perp} \sigma_t \sigma_z \rho_{t,z}. \end{aligned} \quad (7)$$

In Fig. 2 we show the linear correlation coefficients due to eqs. (6) and (5) for emission of negative kaons as a function of the kaon-pair rapidity,  $y_K$ , at  $K_\perp = 1.5$  GeV/c. The results have been obtained for  $Pb + Pb$  at 160 AGeV with a rapidity,  $y_\Delta = 0.90$ , at the edge of the initial central fireball (cf. ref. [12] for more details). Fig. 2 is divided into three ranges of the kaon-pair rapidity,  $y_K \leq -1.4$ ,  $|y_K| < 1.4$ , and  $1.4 \leq y_K$ , which correspond to the freeze-out hypersurface emission zones SWR, NTR, SWR, respectively. SWR denotes the “simple wave region” which is given by the original Riemann rarefaction waves, and NTR denotes the “non-trivial region”, where two rarefaction waves overlap (cf. [14] and also Fig. 3). I.e., kaon pairs with  $|y_K| < 1.4$  are mostly emitted from the NTR, while kaon pairs with  $1.4 < |y_K|$  are mostly emitted from the SWR.

Fig. 2(a) shows that there is an almost perfect linear correlation between the emission times  $t$  and the transverse emission coordinates  $x$  (the longitudinal emission

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<sup>2</sup> Note that in case of a purely Gaussian distribution a non-vanishing  $\rho_{ol}$  is represented by a tilted ellipsoid. However, inspecting Fig. 1 we found a non-ellipsoid shape in the scatter plots. This feature indicates non-Gaussian distributions in real collisions. This will not alter our main conclusion in the text.

coordinates  $z$ ) in the NTR (SWR) of the freeze-out hypersurface. We have  $|\rho_{t,x}| \approx 0.85$  (dotted line) in the NTR, and  $|\rho_{t,z}| \approx 1.0$  (dashed line) in the SWR. For  $|y_K| \rightarrow \infty$  we get  $|\rho_{t,x}| \rightarrow 0$ , and because of symmetry reasons we have  $\rho_{t,z}(y_K = 0) = 0$ . The linear correlation coefficient  $\rho_{x,z}$  (solid line) reaches its extrema at the transition zones of NTR and SWR.

Fig. 2(b) shows that by coincidence  $\rho_{x,z} \approx \rho_{t,x} \cdot \rho_{t,z}$ , but this relation is in general not true. Fig. 2(c) shows that there is a direct proportionality between the (internal) linear correlation coefficient,  $\rho_{x,z}$  (solid line), and the measurable  $\rho_{ol}$  (dashed line):

$$\rho_{ol}(\vec{K}) \propto \rho_{x,z}. \quad (8)$$

Because of this proportionality  $\rho_{ol}$  reflects the strength of linear correlations between transverse and longitudinal emission coordinates of the freeze-out hypersurface. A negative value for  $\rho_{ol}$  at positive average kaon-pair rapidities indicates that for larger longitudinal emission positions, the transverse emission positions become smaller. Conversely, a positive value for  $\rho_{ol}$  at negative average kaon-pair rapidities indicates that for larger longitudinal emission positions (in the negative direction) the transverse emission positions also become smaller. Hence,  $\rho_{ol}$  enables us to measure finite-size effects of fireballs in rapidity space.

In Fig. 3 we show (a) the rapidity spectra,  $dN/dy$ , (b) linear correlation coefficients,  $\rho_{ol}$ , and (c) freeze-out hypersurfaces  $\Sigma(r = 0)$  for emission of negative kaons for  $Pb + Pb$  at 160 AGeV. The results have been obtained with different rapidities,  $y_\Delta$ , at the edge of the initial central fireballs. The parameter  $y_\Delta$  controls the strength of the initial rapidity field of the initial fireball in the hydrodynamical model. A larger value for  $y_\Delta$  results in a larger initial longitudinal rapidity field of the fluid, and subsequently at freeze-out it leads to rapidity spectra of larger widths (*cf.* Fig. 3(a)). Fig. 3(c) shows the corresponding freeze-out hypersurfaces  $\Sigma(r = 0)$ . Consistent with expectation, a larger value for  $y_\Delta$  results in more longitudinally extended freeze-out hypersurfaces. Furthermore, we observe that with increasing  $y_\Delta$ , the NTR of the freeze-out hypersurfaces becomes more and more hyperbola shaped (like the scaling solution for Bjorken initial conditions [15]).

We stress, as shown with Fig. 3(b), that the Full Widths at Half Maximum of the rapidity spectra,  $dN/dy$ , of negative kaons nearly coincide with the positions of extrema of the corresponding linear “out-longitudinal” correlation coefficients,  $\rho_{ol}(\vec{K})$ . We also observe that a more hyperbola shaped freeze-out hypersurface results in smaller absolute values for  $\rho_{ol}$ .

After introducing the linear correlation coefficient,  $\rho_{ol}$ , the difficulties of interpretation encountered above no longer persist, if one uses in case of the generalized Bertsch parameterization the three “radius” parameters,

$R_l$ ,  $R_s$ , and  $R_o$ , and the kinematical quantity  $\rho_{ol}$ . In general, if one analyzes a heavy ion collision which has no azimuthal symmetry, further (measurable) linear correlation coefficients come into existence. In case of the generalized (Gaussian) Bertsch parameterization, one would use for a fit of a two-particle BEC function ( $i, j = l, s, o$ )

$$C(\vec{K}, \vec{q}) = 1 + \lambda(\vec{K}) \times \exp \left[ - \sum_{ij} q_i R_i(\vec{K}) q_j R_j(\vec{K}) \tilde{\rho}_{ij}(\vec{K}) \right], \quad (9)$$

with

$$\tilde{\rho}_{ij}(\vec{K}) \equiv \begin{pmatrix} 1 & -\rho_{os} & -\rho_{ol} \\ -\rho_{os} & 1 & -\rho_{sl} \\ -\rho_{ol} & -\rho_{sl} & 1 \end{pmatrix} \quad (10)$$

$$= - \frac{1}{2 R_i(\vec{K}) R_j(\vec{K})} \left. \frac{\partial^2 C(\vec{K}, \vec{q})}{\partial q_i \partial q_j} \right|_{\vec{q}=0} \quad (11)$$

$$\approx \frac{\sigma_{x_i - \beta_i t, x_j - \beta_j t}}{\sqrt{\sigma_{x_i - \beta_i t}^2 \sigma_{x_j - \beta_j t}^2}}, \quad (12)$$

and  $|\rho_{ij}(\vec{K})| \leq 1 \quad \forall \vec{K}$ . In eq. (12), we have used the notation  $x_l \equiv z$ ,  $x_o \equiv x$ ,  $x_s \equiv y$ ,  $\beta_l \equiv \beta_{||}$ ,  $\beta_o \equiv \beta_{\perp}$ , and  $\beta_s \equiv 0$ .

To summarize, we have investigated the cross term for Gaussian parameterizations of Bose-Einstein correlation functions of purely chaotic hadron sources. We find that at relativistic energies this additional parameter in the so-called Bertsch parameterization can be expressed in terms of a linear “out-longitudinal” correlation coefficient,  $\rho_{ol}(\vec{K})$ , for emission of bosons and the two already known “radius” parameters,  $R_l$  and  $R_o$ . The correlation coefficient provides an additional tool for determining the particle rapidity distributions.

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FIG. 1. Rapidity spectrum of negative pions in arbitrary units for  $S + S$  at 200 AGeV from RQMD (version 2.1). Pion pairs have been formed in different rapidity bins and their corresponding correlations between  $q_o$  and  $q_l$  are shown as scatter plots. The different colors in these plots indicate different relative strengths of the correlation functions (seven equidistant regions). The correlation functions take their maximum values (near to 2) in the center of each scatter plot. The large red areas indicate a minimum correlation strength near to 1.

FIG. 2. Linear correlation coefficients calculated from HYLANDER, for emission of negative kaons for  $Pb + Pb$  at 160 AGeV as a function of the kaon-pair rapidity,  $y_K$ , at  $K_\perp = 1.5$  GeV/c. (a)  $\rho_{t,x}$  (dotted line),  $\rho_{t,z}$  (dashed line),  $\rho_{x,z}$  (solid line); (b) the product  $\rho_{t,x} \cdot \rho_{t,z}$  (dashed-dotted line),  $\rho_{x,z}$  (solid line); (c) the observable  $\rho_{ol}$  (dashed line), and  $\rho_{x,z}$  (solid line).

FIG. 3. (a) Rapidity spectra,  $dN/dy$ , (b) linear correlation coefficients,  $\rho_{ol}$ , at  $K_\perp = 1.5$  GeV/c and (c) freeze-out hypersurfaces  $\Sigma(r = 0)$  for emission of negative kaons for  $Pb + Pb$  at 160 AGeV. The four line styles correspond to four different rapidities,  $y_\Delta$ , at the edge of the initial central fireballs.